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REPLY

Reply to the 'Comment on 'Generalization of the Darboux transformation and generalized harmonic oscillators''

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Abstract

In the comment on our paper, Schulze-Halberg introduces a gauge transformation to gauge away the linear term in momentum. He explicitly shows that the gauge transformation works for n-fold Darboux transformed system. The comment is, however, entirely on the formalism of the Darboux transformation, while most pages of our paper are devoted to the applications of the formalism. In this reply, we add some considerations so that his gauge transformation method can be used to reproduce the solvable Hermitian Hamiltonian systems given in our paper.

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In the comment on our paper [1], Schulze-Halberg introduces a gauge transformation to gauge away the linear term in momentum in the Hamiltonian, then applies the established formalism of Darboux transformation [2, 3] (with an extension to include time-dependent mass). Finally he applies the inverse gauge transformation to find that our formalism is equivalent to the known one. It is well known that the linear term in momentum in a Hamiltonian could be gauged away in one dimension, and it should be done equally well for the n-fold Darboux transformed system, as he explicitly shows. At this point, we thank him for explicitly confirming that our formulae are consistent with the known knowledge [2, 3]. However, it should be mentioned that the comment is entirely on the formalism of the Darboux transformation, while most pages of our paper are devoted to the applications of the formalism. It is also quite true that one can find the solvable systems given in our paper, as the comment may suggest, by first applying the gauge transformation, then using the established Darboux transformation with an extension, and finally applying the inverse gauge transformation, case by case, instead of directly using our formalism stated and proved, relying on the original works [4, 5], in a single printed page of the paper.

On the other hand, it should be noted that some further considerations should be added to the comment, in order that the chain of gauge transformations can be used to reproduce the results in our paper. To obtain the Darboux-transformed Hermitian Hamiltonian systems in our paper, the existence of a purely time-dependent factor $\alpha(t)$ satisfying equation (14) is necessary, for each case of the transformation. In the formalism of the comment, this difficulty lies in the fact that the Hamiltonian of equation (5) is not Hermitian. A modified Hermitian Hamiltonian can be found from equation (5) of the comment, if there exists a purely time-dependent factor $\tilde{\alpha}(t)$ satisfying [1, 3]

$$(\log \tilde{\alpha}_n)_t = -\frac{\mathrm{i}}{2m(t)}\left(\log \frac{W_n}{\overline{W}_n}\right)_{xx} \tag{1}$$

in the notations of the comment, where $\overline{W}_n$ denotes the complex conjugate of W_n. In this case, equation (5) of the comment can be rewritten as

$$\mathrm{i}\chi_t^D + \frac{1}{2m}\chi_{xx}^D + \left(2\left(m\int R\,\mathrm{d}x\right)_t + 2mR^2 - V_0 + \frac{1}{2m}(\log(W_n\overline{W}_n))_{xx}\right)\chi^D = 0. \tag{2}$$

Equation (2) can be interpreted as a time-dependent Schrödinger equation of the Hermitian Hamiltonian

$$H_n = \frac{p^2}{2m} + V_0 - 2\left(m\int R\,\mathrm{d}x\right)_t - 2mR^2 - \frac{1}{2m}(\log(W_n\overline{W}_n))_{xx}, \tag{3}$$

for any real R. A solution of equation (2) is now written as

$$\chi^D = \tilde{\alpha}_n(t)\chi = \tilde{\alpha}_n(t)\frac{W_{n,\Phi}}{W_n} \tag{4}$$

which shows that, if W_n has any zero in the entire space of x, the Darboux transformed wavefunctions may not be square-integrable, so that square-integrablity must be checked in each case of the applications. In all the applications in our paper [1], $R(x,t)$ is given as

$$R(x,t) = -a(t)x, \tag{5}$$

and thus T may be written as

$$T = \exp(-\mathrm{i}m(t)a(t)x^2). \tag{6}$$

This operator of the gauge transformation guarantees that the difficulties mentioned above can be solved for the system of the Hamiltonian in equation (3) in all applications of our paper [1]: from the relation between W_n and $\hat{W}_n$ in the comment, one may find that $\tilde{\alpha}_n(t)$ can be equal to $\alpha_n(t)$ in each case. Further, from the gauge transformation, one can see that there is no zero in W_n, as $\hat{W}_n$ does not vanish in the entire space of x in the applications.

Indeed, the quadratic system with or without an inverse-square interaction has been obtained though a sequence of the unitary transformations [6, 7], and the gauge transformation is a subclass of the unitary transformations [6]. Instead of using the gauge transformation, one may, therefore, directly find the solutions to equation (2), χ^D, by choosing $a(t) = 0$, and modifying $c(t)$ accordingly in the formulae of sections 3 and 4 of our paper [1]. Last, but not least, it should be mentioned that the results obtained in this way are still new over the known results [2, 3], as the shape difference between the potentials of the original system and of the Darboux transformed system could oscillate harmonically. Though other (conditionally) exactly solvable systems have been found from the simple harmonic oscillator system through the supersymmetric method [8], the solvable systems of the shape difference oscillating harmonically have been first given in our paper [1].

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